M. Abouzaid - 28/1/13 - Dilations, equivariance & formality in sympl. geometry I

<u>Mirror symmetry, 0^{th} approximation</u> ("without corrections")

M symplectic $\rightsquigarrow$ $\mathcal{L} = \{\text{embedded Lagr. submflds}\}/\text{Ham. isotopy}$

Local deform^s of L = graph of closed 1-forms, mod out by exact 1-forms

$\Rightarrow$ $\mathcal{L}$ is locally modelled on $H^1(L,\mathbb{Z})\otimes\mathbb{R}$

$T\mathcal{L} \supset$ canonical lattice $\rightsquigarrow$ <u>integral affine structure</u>

$\mathcal{L}^{\mathbb{C}}$ complexification: pairs (L,∇), (∇ $U(1)$ local system) loc. modelled on $H^1(L,\mathbb{Z})\otimes\mathbb{C}$

carries a canonical quadratic volume form (since holonomy $GL(n,\mathbb{Z})$)

Mirror symmetry: find scheme Y (possibly analytic, noncomm., derived, ...)

s.t. "moduli of sheaves" on Y corresponds to $\mathcal{L}^{\mathbb{C}}$.

Always have $2c_1(Y) = 0$ (since $Y = \{\text{skyscraper sheaves of pts}\} \subset \mathcal{L}^{\mathbb{C}}$).

<u>Example:</u> $M = T^2$ with area A. $\mathcal{L} = \coprod_{\substack{\alpha\in H_1(T^2,\mathbb{Z})\\ \text{primitive}}} \mathcal{L}_\alpha$, $\mathcal{L}_\alpha \cong \mathbb{R}/A\cdot\mathbb{Z}$

area mod A

$\rightsquigarrow$ $\mathcal{L}^{\mathbb{C}}_\alpha = \mathbb{R}/A\mathbb{Z} \times i\mathbb{R}/2\pi i\mathbb{Z}$

(is in fact the mirror!)

<u>Observe:</u> in $M = T^2$ case there's an $S^1\times S^1$-action on the mirror

- real action of S^1 $\longleftrightarrow$ symplectomorphisms of M (translations)
- imaginary action $\longleftrightarrow$ $\otimes$ local system on M.

ie. $(L,\eta) \mapsto (L, \eta\otimes\alpha_{|L})$

for α loc. sys. on M.

At <u>infinitesimal level:</u>

$\mathbb{R}$-action on mirror Y $\longleftrightarrow$ holomorphic vector field $\in H^0(Y,T_Y)$

$\longleftrightarrow$ deformation of the diagonal in $Y\times Y$

$$H^0(Y,T_Y) \subset \mathrm{Ext}^1_{Y\times Y}(\mathcal{O}_\Delta,\mathcal{O}_\Delta) = H^0(Y,T_Y)\oplus H^1(Y,\mathcal{O})$$

Under HMS, $\mathrm{Ext}^1(\mathcal{O}_\Delta,\mathcal{O}_\Delta) \simeq HF^1(\Delta_M,\Delta_M)$ Lagr. in $M\times\overline{M}$.

If M is closed, $HF^1(\Delta,\Delta) \simeq H^1(M,\mathbb{C})$
& no instantons

$$\simeq \underset{\substack{\text{tangent space}\\ \text{to } Symp(M)/Ham}}{H^1(M,\mathbb{R})} \oplus \underset{\substack{\text{tangent space to}\\ \text{local systems on } M}}{H^1(M,i\mathbb{R})}$$

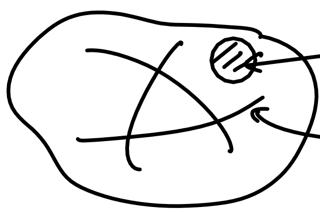
More generally, allow M open (eg affine $\subset \mathbb{C}^N$, ..)

<u>Mirror symm. with corrections:</u>

Now define $\mathcal{L}^{\mathbb{C}}$ = space of pairs (L,b), $b \in H^1(L,\mathbb{C})$ satisfying Maurer-Cartan equation up to equivalence in the Fukaya category

Kontsevich-Soibelman, ...: locally patches $\simeq$ Floer cohomology $HF^1(L,b)$
Often this is $\simeq H^1(L,\mathbb{C})$ (<u>but not canonically</u>)

$\rightsquigarrow$ chambers

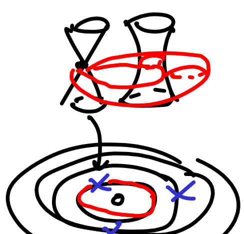


in chamber, $\simeq (L,\nabla)$

walls coming from holom. discs w/ boundary on L

* Reglue the complements of the walls by nontrivial gluings that depend on enumerative geom. of discs.

<u>Ex</u>: (AAK): $M = A_k$ Milnor fibre - conic
$= \{ x^2+y^2 = p(z) \mid z \neq 0 \} \subset \mathbb{C}^3$



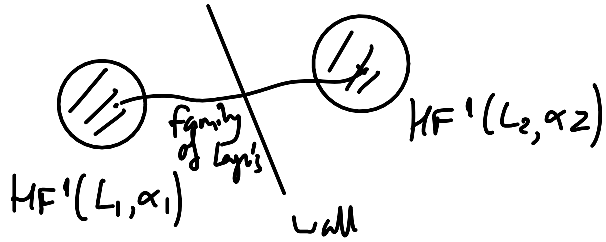
$p(0) \neq 0$, $\deg p = k+1$, all roots simple.

Construct a Lagr.-T^2 fibration (by fixing "height" on fibers)

$$\begin{array}{ccc} T^2 & \subset & M \\ \downarrow & & \downarrow z \\ S^1 & \subset & \mathbb{C}^* \end{array}$$

If $S^1 \subset \mathbb{C}^*$ passes through a zero of p, then get $\begin{cases} T^2 & \text{if height} \neq 0 \\ \text{immersed } S^2 & \text{if height} = 0. \end{cases}$

... $\rightsquigarrow$ cut $\mathcal{L}^{\mathbb{C}}$ open along the walls & glue via K-S.



Explicit answer for mirror of $A_k^{\text{Milnor}} \setminus \text{conic} \subset \mathbb{C}^* \times \mathbb{C}^2$:

i.e. let $A_k^{\text{sing}} \setminus \text{conic} = \{ z^{k+1} = x^2 + y^2 \,/\, z \neq 1 \} \subset \mathbb{C}^* \times \mathbb{C}^2$

and the mirror Y is its resolution.

Note: Y is actually diffeo to M in this example

($Y \supset$ chain of k $\mathbb{P}^1$'s whereas $M \supset$ chain of Lag. S^2's)

Mystery: $H^0(Y, T_Y) \oplus H^1(Y, \mathcal{O}_Y)$ has infinite rank

vs. $H^1(M, \mathbb{C})$ has rank 1.

(eg. for case $k = 0$: $M = Y = \mathbb{C}^2 \setminus \text{conic}$)

Answer: $\mathrm{Ext}^1(\Delta_Y, \Delta_Y) = H^0(Y, T_Y) \oplus H^1(Y, \mathcal{O}_Y)$

$\simeq SH^1(M, \mathbb{C})$ symplectic cohomology

We'll try to analyze some of the geometric meaning in the next talks - eg. dilations wrt the volume form.